How Mathematical Proof Works: An Introduction

- Goals: discover, write, check, and use proofs across mathematics
- Key distinction: exploring why a claim holds vs. proving it must
- For high school and early university students, adult learners, and teachers
- Roadmap: quantifiers, direct and indirect proof, existence, induction, writing, logic, formal verification
- Preview: the sum of the first n odd numbers equals n squared



Why Proof Matters: Evidence, Reasoning, and Curriculum Expectations

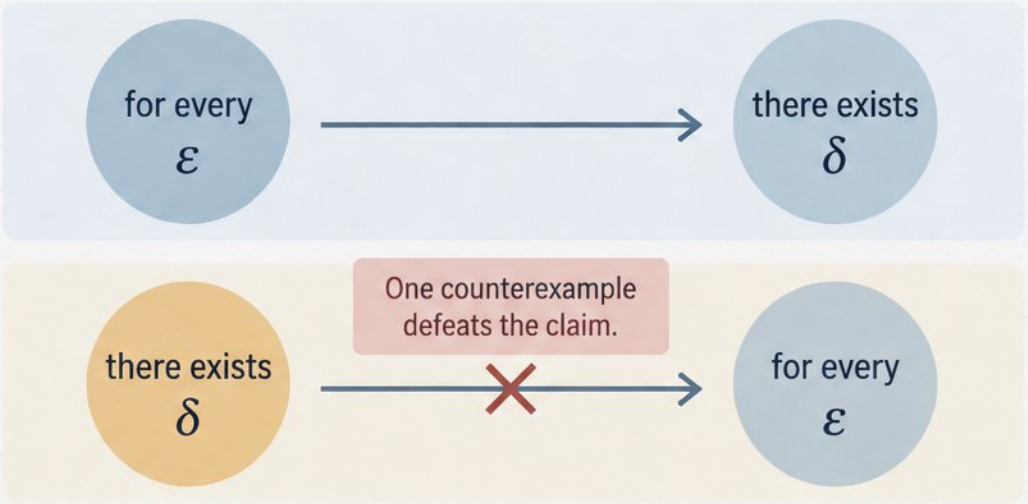
- Checking many cases supports a conjecture but never proves it
- A proof is a reasoned chain from assumptions and definitions
- Common Core MP3: construct viable arguments and critique others' reasoning
- IB: proof builds critical thinking; Cambridge: structured, checkable reasoning
- Learners arrive with uneven proof preparation across curricula



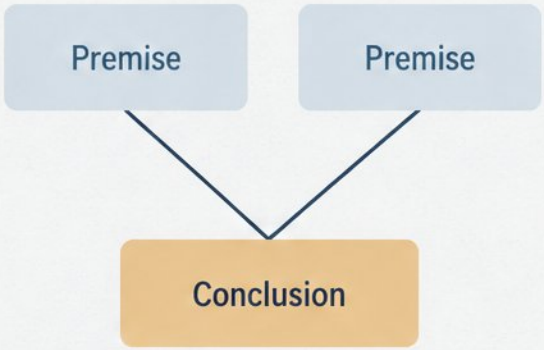
Statements, Quantifiers, and Truth Conditions

- Statements have definite truth values; open sentences need variables bound.
- One counterexample defeats a universal claim; one witness establishes existential.
- Quantifier order matters: ‘for every ε, there exists δ’ differs from its reverse.
- Definitions fix meaning precisely and support proof.
- Label antecedent and consequent; distinguish necessary, sufficient, equivalent conditions.

QUANTIFIER ORDER MATTERS



FROM PREMISES TO CONCLUSION



CONDITIONS

Antecedent:	If …
Consequent:	then …
Necessary:	A is necessary for B.
Sufficient:	A is sufficient for B.
Equivalent:	A is equivalent to B.

Direct Proof and Proof by Cases

- Direct proof: assume hypotheses, unpack definitions, derive conclusion step by step
- Example: if n is even, then n^2 is even — clean algebra
- Proof by cases: partition exhaustively, e.g., parity or sign
- Working backward from the conclusion is scratch work, not proof
- Pitfalls: assuming the claim; manipulating both sides unjustified

Example: if n is even, then n^2 is even

Case 1: n is even

Let $n = 2k$.

Then $n^2 = (2k)^2$

$= 4k^2$

$= 2(2k^2)$.

So n^2 is even.

Case 2: n is odd

Let $n = 2k + 1$.

Then $n^2 = (2k+1)^2$

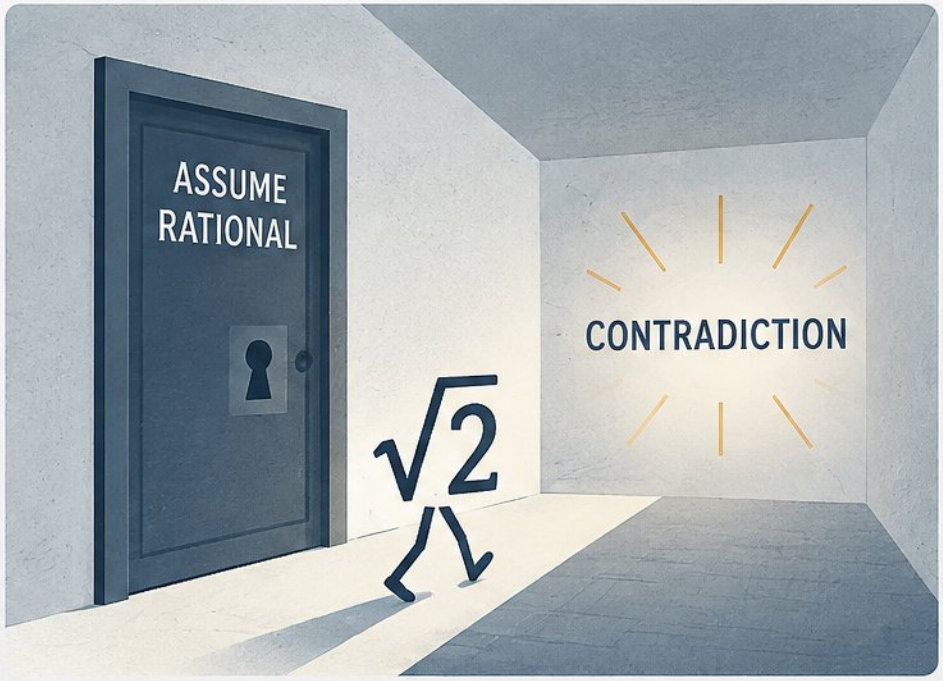
$= 4k^2 + 4k + 1$

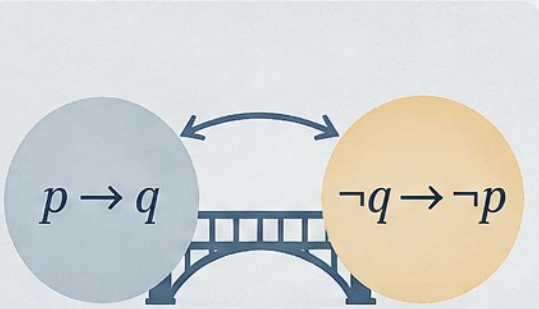
$= 2(2k^2 + 2k) + 1$.

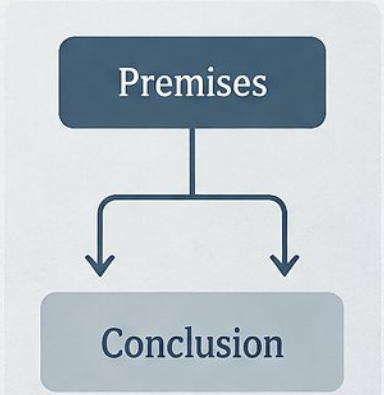
So n^2 is even.

Indirect Reasoning: Contrapositive and Contradiction

- Statement and contrapositive are logically equivalent; converse is not
- Contradiction: assume the negation, derive an impossibility
- Classic: $\sqrt{2}$ is irrational — use the definition of rational
- Contrapositive proof proves the implied statement directly
- Warning: negate statements carefully; infinitude of primes example







Existence, Construction, and Counterexample

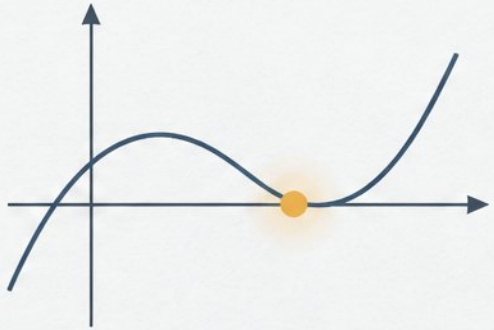
- **Constructive existence:** give an explicit example or algorithm as witness
- **Nonconstructive existence:** prove an object exists without computing it, e.g., a root
- **Counterexample:** one verified case disproves a universal claim
- **Uniqueness:** prove existence plus at most one candidate
- **Structure:** existence and uniqueness proofs on small integers

Constructed Witness



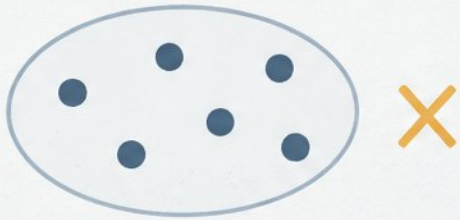
Give an explicit example or algorithm as witness.

Located but Not Computed



Prove an object exists without computing it, e.g., a root.

A Single Counterexample

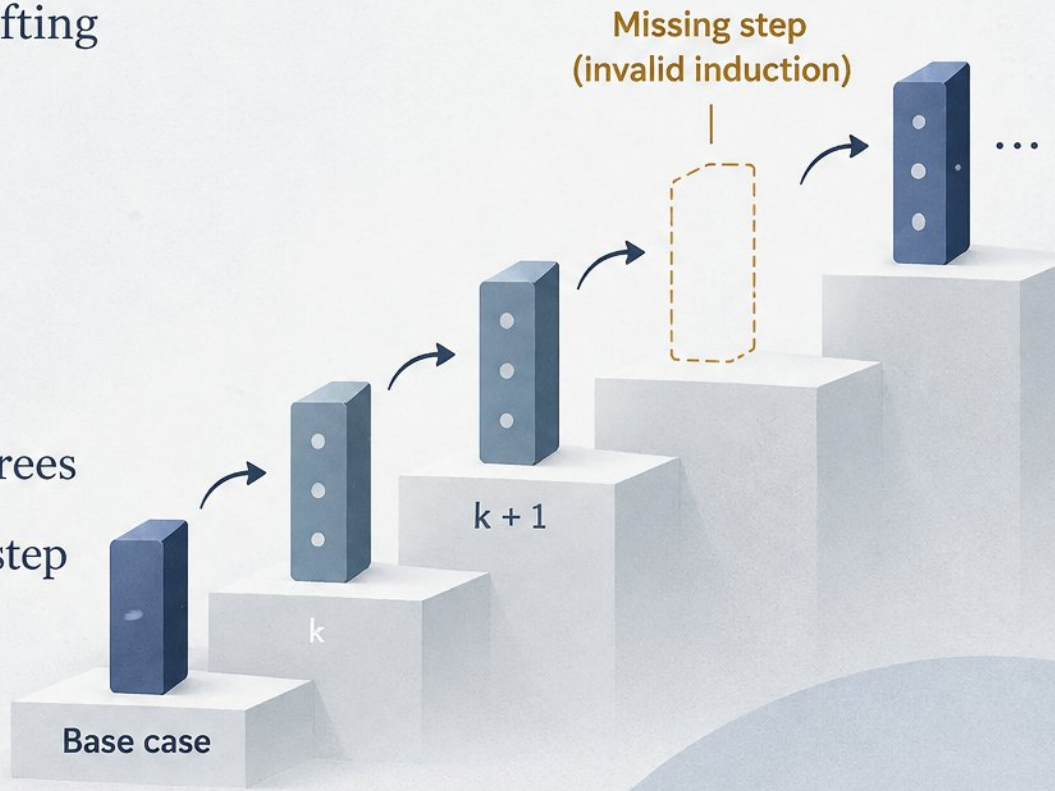


One verified case disproves a universal claim.

Uniqueness: prove existence plus at most one candidate.

Mathematical Induction and Recursion

- Base case and inductive step; the induction hypothesis does the heavy lifting
- Example: $1 + 3 + \ldots + (2n - 1) = n^2$, with an algebraic inductive step
- Strong induction: the hypothesis may cover all earlier cases
- Recursive definitions and structural induction for sequences, expressions, trees
- Invalid induction: gaps in the inductive step or a skipped base case



Example: $1 + 3 + \ldots + (2n - 1) = n^2$

Base case $(n = 1)$: $1 = 1^2$

Inductive step: assume $1 + 3 + \ldots + (2k - 1) = k^2$

Then $1 + 3 + \ldots + (2k - 1) + (2(k + 1) - 1)$

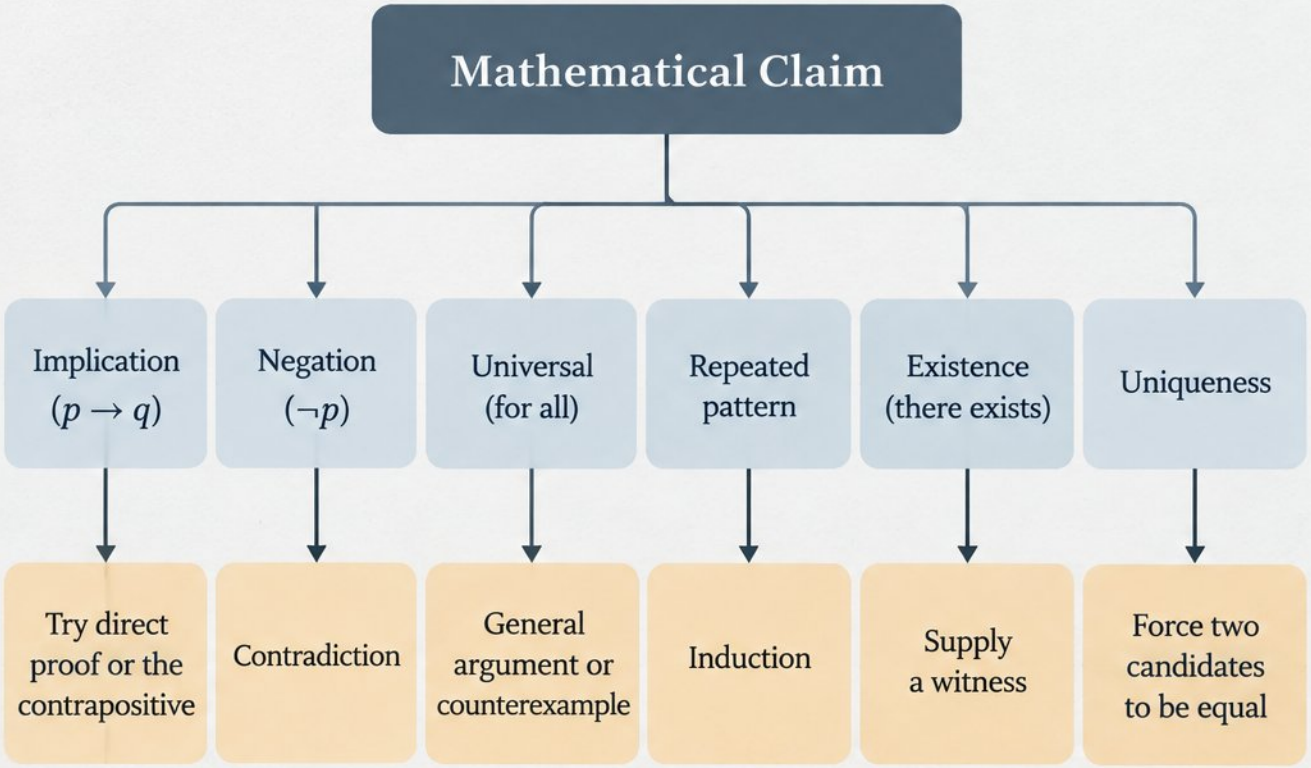
$= k^2 + (2k + 1) = (k + 1)^2$

Hence $1 + 3 + \ldots + (2n - 1) = n^2$ for all n.

A SMALL STEP BUILDS A BIGGER UNDERSTANDING

Choosing a Proof Strategy

CLAIMS
IDEAS
PROOFS
UNDERSTANDING



Choosing a Proof Strategy

- Implication: try direct proof or the contrapositive
- Negation: contradiction; universals: general argument or counterexample
- Repeated pattern: induction; existence: supply a witness
- Uniqueness: force two candidates to be equal
- Sketch first, then check definitions, hypotheses, conclusion, quantifiers

Reading and Writing Proofs Like a Mathematician

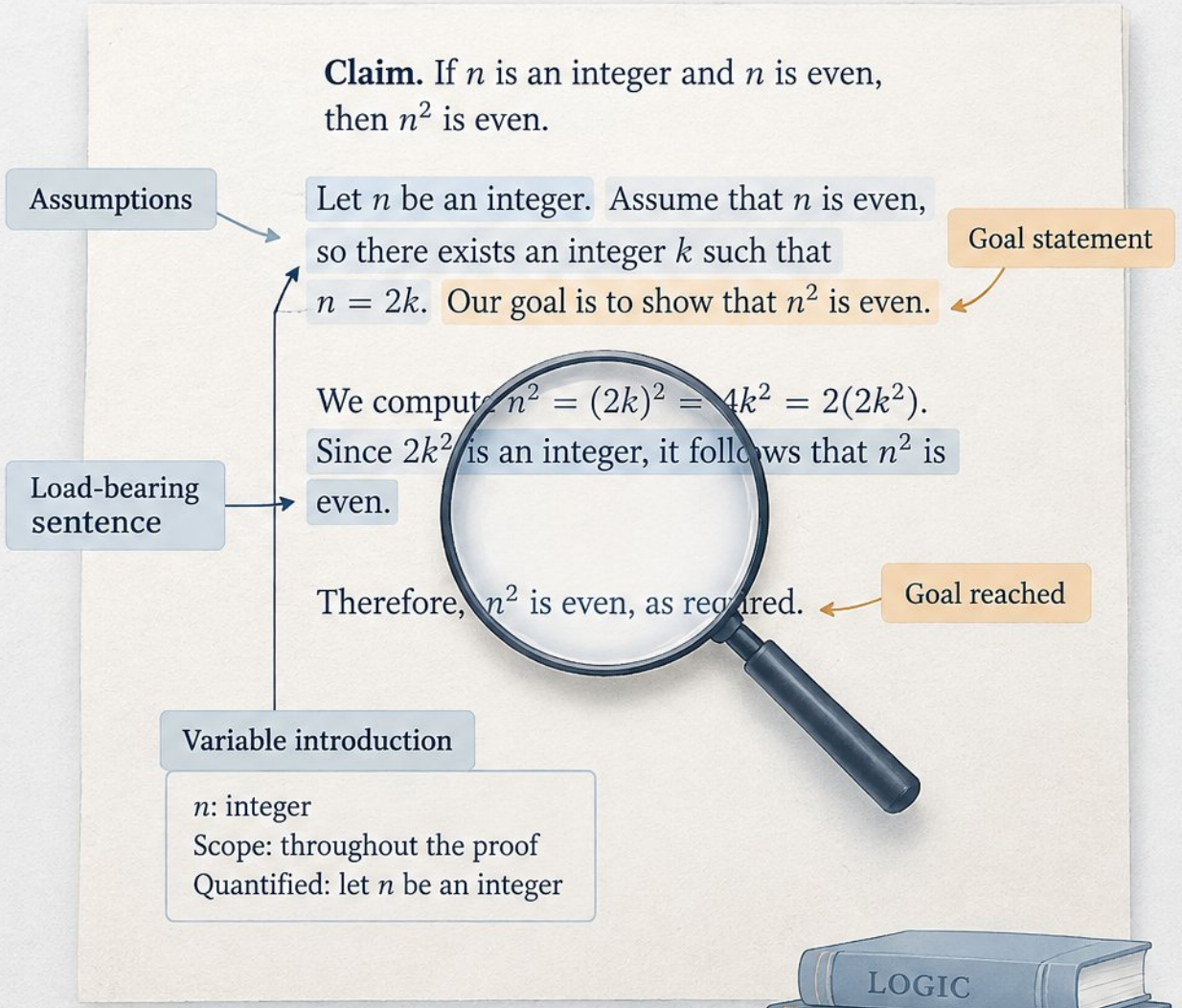
Structure: introduce assumptions, state the goal, give reasoning, confirm the goal is reached

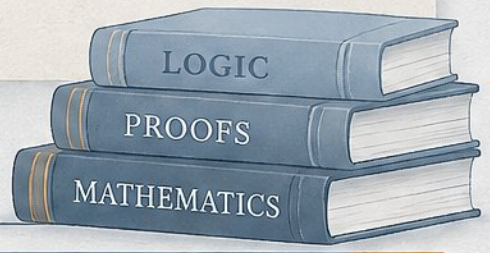
Every sentence should be load-bearing; cut unrelated facts and digressions

Introduce each variable: its type, scope, and how it is quantified

Write complete sentences and paragraphs; avoid ∀ and ∃ symbols in prose

Active reading: summarize the claim, question each step, then reconstruct from memory





Proof Styles That Communicate Clearly

- Proofreading rule: an outside math friend follows every implication easily
- Respect the equal sign; use "if…then…" constructions deliberately
- Know your audience: peers for coursework, more detail for newcomers
- Draft on scrap paper first, then write in logical order
- Compare a weak proof with its corrected version side by side

A Weak Proof

Assume \n is even.

Then n^2 is even.

Also, $n = 2k$.

So $n^2 = 4k^2$.

Hence n^2 is even.

Issues: key implication is not justified, steps are out of order, and the logic is unclear.

A Corrected Proof

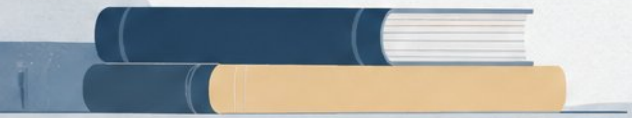
Assume n is even.

Then $n = 2k$ for some integer k.

So $n^2 = (2k)^2 = 4k^2$.

Since $4k^2$ is even, it follows that n^2 is even.

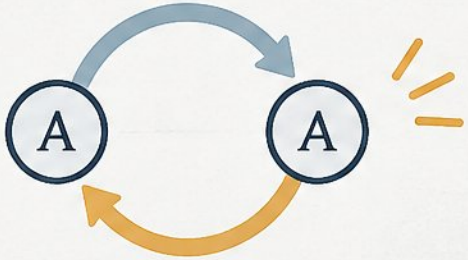
Each implication is justified and follows in a logical order.



CLEARER REASONING
BRIGHTER MATHEMATICS

Verifying Logic: Common Errors and Fallacies

Circular reasoning



Circular reasoning: assuming the conclusion you are trying to prove

Invalid implication moves



Affirming the consequent and denying the antecedent: invalid implication moves

Quantifier and negation slips



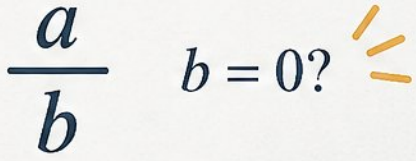
Quantifier and negation slips: treating a theorem as its converse

Edge cases that break proofs



Edge cases that break proofs: zero, negatives, the empty set, boundaries

Hidden assumptions



Hidden assumptions: unproved background facts or dividing by a possibly zero quantity

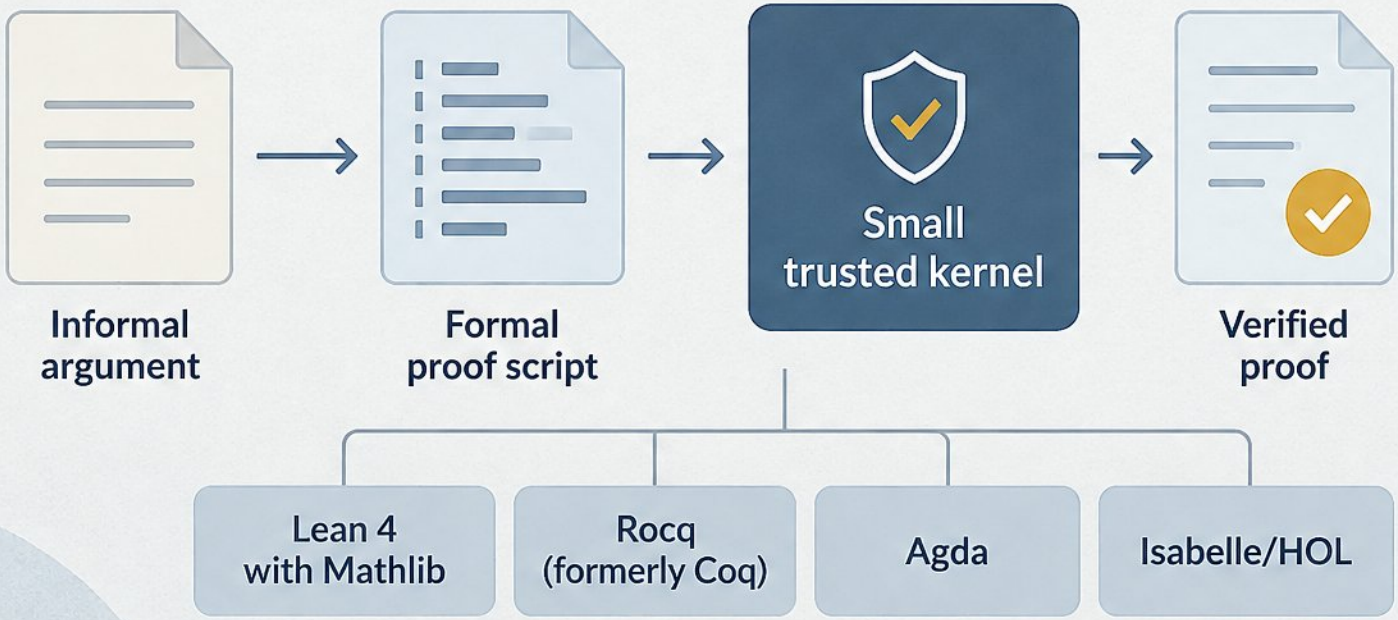
Workshop: Practice and Proof Audit

- Guided exercises cycling through direct, indirect, and inductive techniques
- Peer review: locate a gap in a supplied proof and repair it
- Timed writing task: produce a complete direct or inductive proof
- Reflection: which proof technique fits which claim shape
- Debrief linking observed errors to earlier lessons; rubric stresses logical completeness and clarity



Formal Proof and Proof Assistants

- Informal pen-and-paper proofs become formal, machine-checked proofs
- A proof assistant checks proofs against precise logic via a small trusted kernel
- Lean 4 with Mathlib, Rocq (formerly Coq), Agda, Isabelle/HOL
- Verified real systems: CompCert C compiler, seL4 microkernel
- Formal proofs clarify what counts as a valid definition or proof



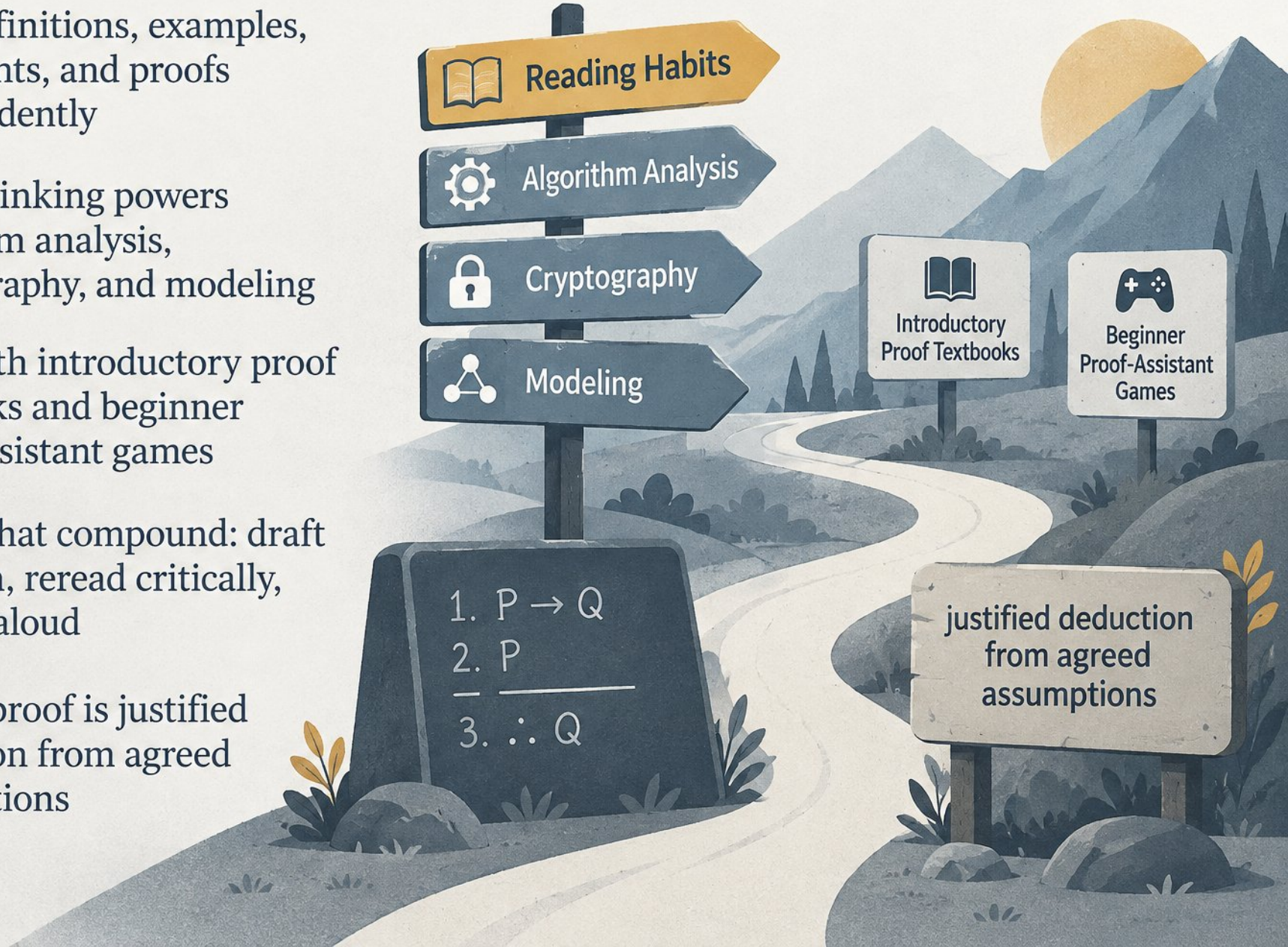
Verified real systems

CompCert C compiler

seL4 microkernel

Next Steps: Habits and Resources for Continued Practice

- Read definitions, examples, statements, and proofs independently
- Proof thinking powers algorithm analysis, cryptography, and modeling
- Start with introductory proof textbooks and beginner proof-assistant games
- Habits that compound: draft in rough, reread critically, explain aloud
- Recap: proof is justified deduction from agreed assumptions



PersonWise.

PersonWise • AI digital-human course platform

Scan the code or click anywhere to watch this course live, with voice Q&A

personwise.ai/t/af2a390c/public