Sizes of Infinity:
Countable and Uncountable

Hook:
some infinities are bigger than others —
here is what that means

Roadmap:
matching → Hilbert's Hotel →
diagonalization → the continuum →
the tower of infinities → CH

Core example:
naturals can be listed, reals cannot
(Cantor 1874; diagonal 1891)

No formal proofs needed:
concrete examples, tables, plain-language
reasoning throughout



Counting Without Counting: Matching as the Universal Tool

- A bijection pairs every element of one set with exactly one of another.
- Warm-up: pair seats with people, shoes with shoelaces — count without reciting numbers.
- Definition: two sets have the same cardinality exactly when a bijection exists between them.
- Matching, not listing, is the right definition — it extends cleanly to infinite sets.
- Flag early: finite intuition (proper subsets are smaller) breaks for infinite sets.





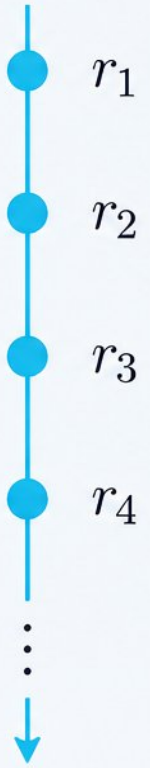
Countably Infinite: When Everything Can Be Listed

- **Countably infinite:** elements can be listed and matched one-to-one with $\mathbb{N}$
- Surprising examples: integers, positive rationals, pairs of naturals, finite strings
- **Integers:** $0, 1, -1, 2, -2, \ldots$
- Positive rationals: sweep diagonals of the numerator–denominator grjd
- Dense yet listable: between any two rationals there is another

Sweep the numerator–demoninator grid

	1	2	3	4	⋯
1	$\frac{1}{1}$	$\frac{1}{2}$	$\frac{1}{3}$	$\frac{1}{4}$	⋯
2	$\frac{2}{1}$	$\frac{2}{2}$	$\frac{2}{3}$	$\frac{2}{4}$	⋯
3	$\frac{3}{1}$	$\frac{3}{2}$	$\frac{3}{3}$	$\frac{3}{4}$	⋯
4	$\frac{4}{1}$	$\frac{4}{2}$	$\frac{4}{3}$	$\frac{4}{4}$	⋯
⋯	⋯	⋯	⋯	⋯	⋯

A single list



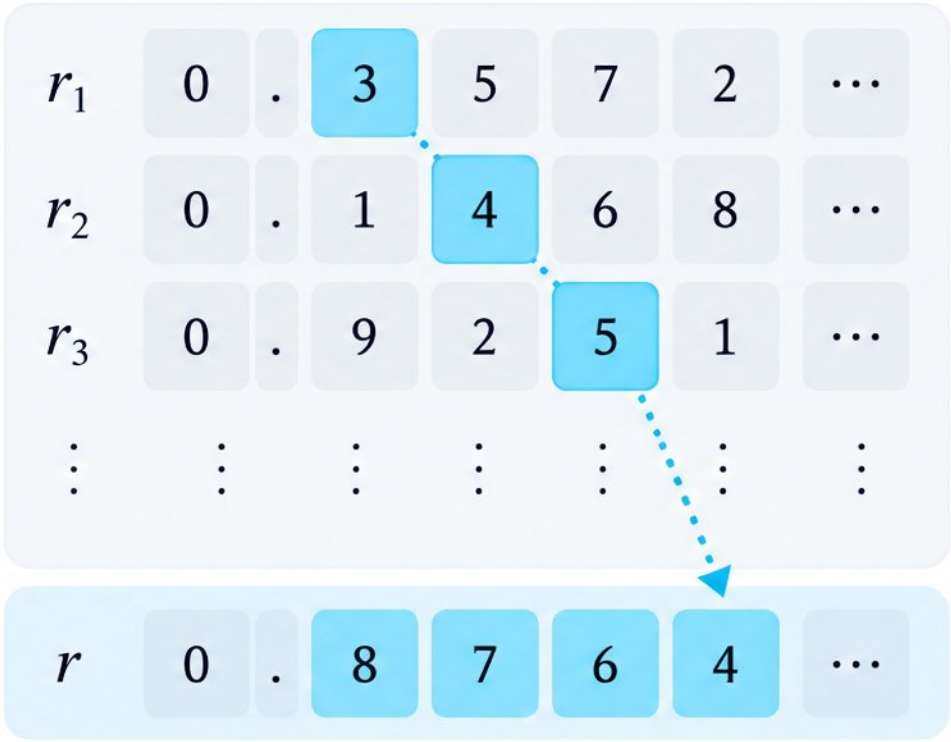
Hilbert's Hotel: Infinity Always Has Room

- A hotel with rooms 1, 2, 3, … is full — yet it still has vacancies.
- One new guest: occupant of room n moves to $n + 1$; room 1 frees up.
- Infinitely many guests: move each occupant to room $2n$; odd rooms open.
- Countably many buses of countably many guests: enumerate pairs (bus, seat).
- A countable union of countable sets is countable.
- Guests indexed by all infinite 0–1 sequences could never be accommodated.



Diagonalization: Building a Number That Escapes Any List

- Assume, for contradiction, every real in (0, 1) is listed: $r_1, r_2, r_3, \ldots$
- Write each r_n as an infinite decimal; walk down the diagonal
- Build r whose nth digit differs from the nth digit of r_n
- Then r differs from every listed number in at least one place
- Avoid digits 0 and 9, so 0.999… = 1.000… cannot hide a match
- Takeaway: no listing can be complete — not just this one



The Reals Cannot Be Listed: The Uncountable Continuum

$\mathbb{N}$

A countable list



$$|\mathbb{N}| = \aleph_0$$

$\mathbb{R}$

A densely filled line



$$|\mathbb{R}| = \mathfrak{c}$$

- No bijection exists between $\mathbb{N}$ and $\mathbb{R}$: the reals are uncountable
- $\aleph_0$ (aleph-null) is the size of $\mathbb{N}$; $\mathfrak{c}$ (the continuum) is the size of $\mathbb{R}$, with $\aleph_0 < \mathfrak{c}$
- Same size, different look: the interval (a, b), the whole line $\mathbb{R}$, and even $\mathbb{R}^n$ all have cardinality $\mathfrak{c}$
- Cantor's 1874 nested-interval proof and his 1891 diagonal argument reach the same result by different routes
- Sets of size $\mathfrak{c}$ include the irrationals, the transcendentals, and the "thin" Cantor set

An Endless Tower:
Power Sets and Cantor's Theorem

- Cantor's theorem: the power set of any set A is strictly larger than A — $|A| < 2^{|A|}$.
- Diagonal proof: pair each element a with a subset S(a); define $D = \{ a : a \notin S(a) \}$.
- D escapes every S(a) — just as the diagonal number escapes every listed real.
- So no largest infinity exists: $\aleph_0$, $2^{\aleph_0}$, $2^{(2^{\aleph_0})}$, ... climb forever.
- Notation: $2^{\aleph_0}$ is exactly the continuum $\mathfrak{c}$; the beth numbers name this ladder.
- Takeaway: infinity is not one destination but a whole hierarchy of sizes.



Cardinal Arithmetic:
When Infinity Plus Infinity Changes Nothing

$\aleph_0 + \aleph_0 = \aleph_0$



Hilbert's Hotel
in arithmetic form

$\aleph_0 \times \aleph_0 = \aleph_0$



$2^{\aleph_0} > \aleph_0$



the content
of uncountability

$\mathfrak{C} \times \mathfrak{C} = \mathfrak{C}$

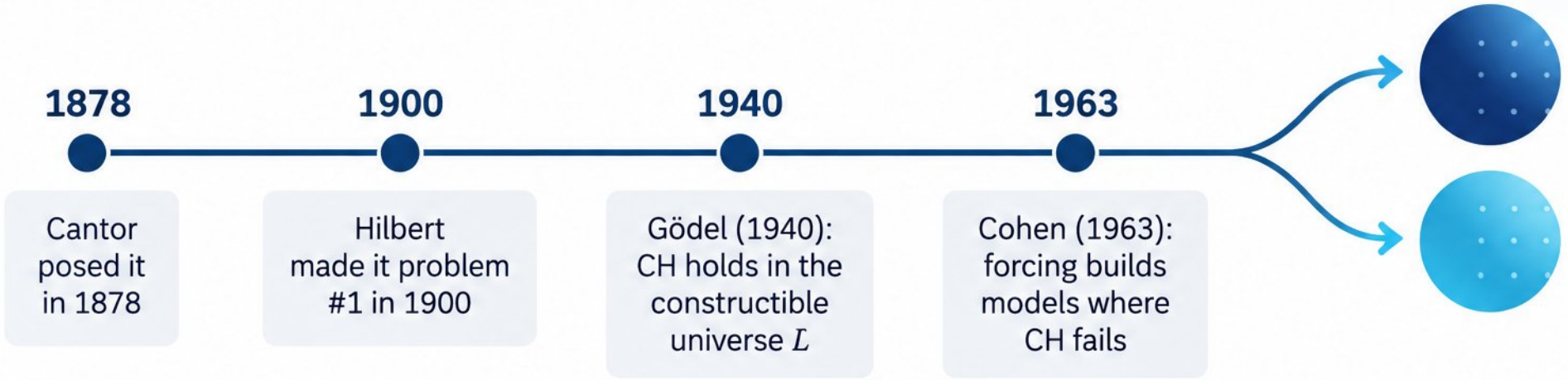


matching $|\mathbb{R}^2| = |\mathbb{R}|$

4 | Continuous functions $\mathbb{R}\to\mathbb{R}$: only $\mathfrak{C}$ many; all functions $\mathbb{R}\to\mathbb{R}$: a larger $2^{\mathfrak{C}}$

5 | Keep order type (ω) separate from size ($\aleph_0$) — same set, different concepts

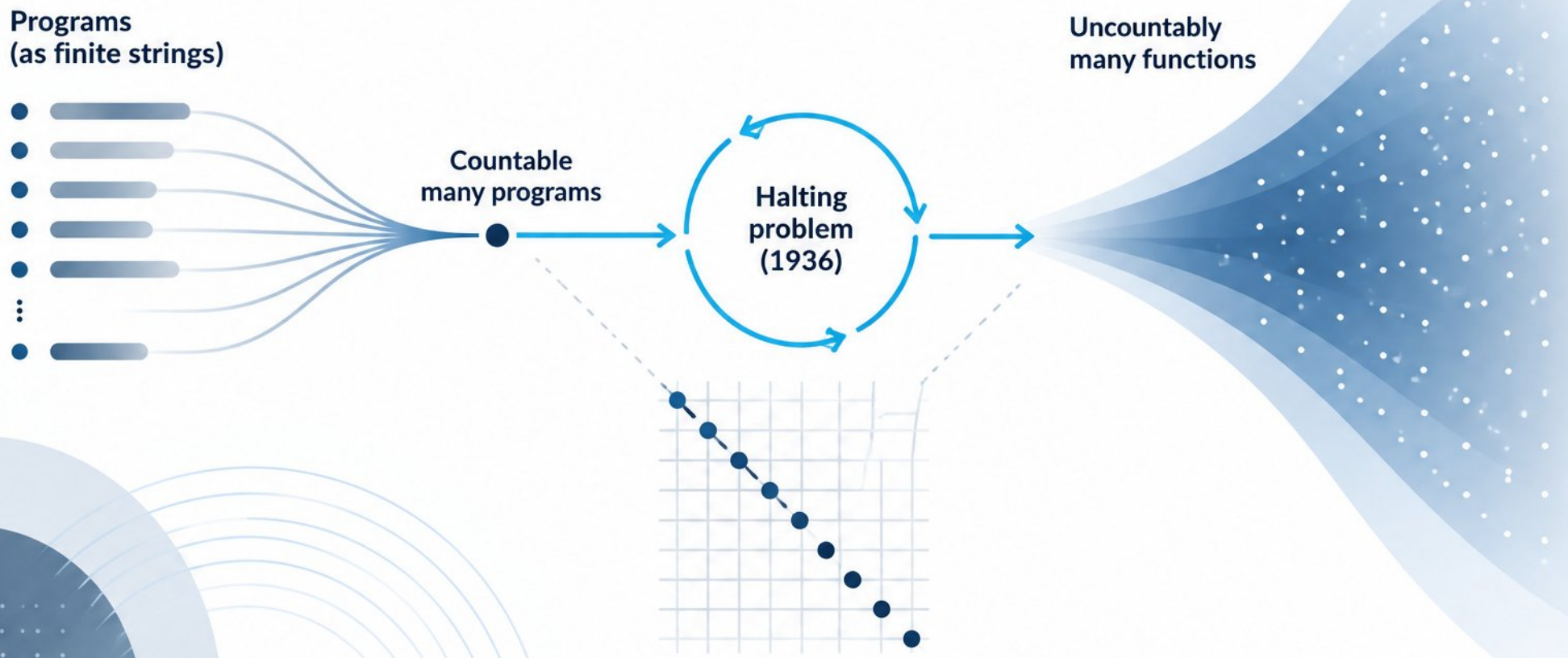
Is There Anything in Between? The Continuum Hypothesis



- CH asks: is there a size strictly between $\aleph_0$ and $\mathfrak{c}$?
- Equivalently, does $2^{\aleph_0} = \aleph_1$?
- Cantor posed it in 1878; Hilbert made it problem #1 in 1900
- Gödel (1940): CH holds in the constructible universe L — so ZFC cannot disprove it
- Cohen (1963): forcing builds models where CH fails — so ZFC cannot prove it
- Verdict: CH is independent of ZFC, like the parallel postulate in geometry

Why It Matters: Computation, Logic, and the Real World

- Countably many programs, uncountably many functions: most are uncomputable
- Turing's halting problem (1936) is diagonalization in disguise
- Gödel's incompleteness theorem uses the same diagonal technique
- Floating-point numbers form a countable sliver of the continuum
- Discovering bigger infinities was a genuine mathematical discovery



Common Traps and Quick Exercises

Myth

Myth: "infinity is a number" — $\aleph_0$ and $\mathfrak{c}$ are sizes of sets.

Myth

Myth: "all infinite sets are the same size" — $\mathbb{N}$ and $\mathbb{R}$ are both infinite but not equinumerous.

Myth

Myth: "you cannot compare infinities" — bijections give a rigorous, well-defined ordering.

Exercise

Exercise: Are the finite subsets of $\mathbb{N}$ countable? Yes — group by maximum element.

Exercise

Exercise: Are the real algebraic numbers countable? Yes — order polynomials by height and list roots.

Exercise

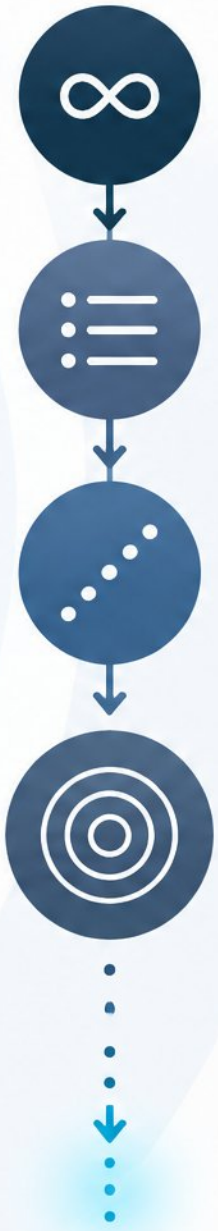
Exercise: Is the Cantor set uncountable? Yes — bijection with [0, 1] via base-3/base-2 digits.

Where Set Theory Is Today

- CH's independence opened the field, not closed it
- Three axiom families: large cardinals, forcing axioms, determinacy
- Levy–Solovay: large cardinals alone cannot settle CH
- Active frontier: inner models, HOD, descriptive set theory
- Ultimate-L and Sealing: competing visions for new axioms
- Applications reach topology, algebra, and operator algebras



Recap: Matching, Listing, Diagonalization



- **Size means matching:** a bijection pairs every element, nothing left over
- **Countably infinite:** $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$, pairs, and finite strings can all be listed
- **Diagonalization escapes any list:** the reals are uncountable, so $\aleph_0 < \mathfrak{c}$
- **Power sets climb forever:** $\aleph_0$, $2^{\aleph_0}$, $2^{(2^{\aleph_0})}$, … — no largest infinity
- **CH (is anything between $\aleph_0$ and $\mathfrak{c}$?)** is independent of ZFC (Gödel 1940; Cohen 1963)
- **Takeaway:** every infinity can be outgrown

Close: Q&A and Where to Go Next

- Which surprised you more: listing the rationals, or the unlistable reals?
- Does CH's independence change what "as many reals as..." means?
- Go deeper: set theory, computability (diagonalization, undecidability), real analysis
- Read Cantor's 1874 paper; overviews of Gödel (1940) and Cohen (1963)
- Matching, listing, diagonalizing: tools that reach far beyond set theory

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